# The Equilibrium Charge Distribution of Conducting Dust Aggregates, Found Through a Potential-Conscious Approach

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***Abstract*—We have approximately solved the problem of finding the equilibrium charge distribution on an aggregate of conducting dust grains in a dusty plasma. Using the assumptions that the grains are spherical and have spherically symmetric charge distributions regardless of the presence of other charged grains, we designed a program to calculate the amount of charge that each grain in an aggregate should have, given an initial amount of total charge present on the aggregate. Our results agree with theoretical predictions of how the charge distribution on a conducting aggregate should behave; that is, the charge distribution is biased toward those aggregates that lie at the extreme points on the aggregate.**

***Index Terms*—Aggregate, Charge Distribution, Conducting, Dusty Plasma.**

## I. Introduction

THEORETICAL research concerning dusty plasmas began to receive a large amount of attention in the 1980s, when it was discovered that the structure of Saturn's rings could not be explained as a result of simple gravitational interactions. Dusty plasma research has since become relevant to several other areas, such as the production of silicon wafers and the formation of planetesimals. Dusty plasmas commonly possess vastly different properties from ordinary plasmas. The immersed dust particles tend to become charged by various means and form aggregates, causing unusual phenomena to be exhibited, such as otherwise impossible waves and variations in the local density of the plasma.

In this paper, we concern ourselves with our research on the distribution of charge on an aggregate of conducting dust grains. The charge distribution of an aggregate in a dusty plasma determines the electric dipole moment of the aggregate, and therefore affects the torque that the aggregate experiences as it travels through the plasma. This torque inevitably determines the behavior of collisions between this aggregate and other aggregates in the plasma, and so the charge distribution on an aggregate plays in important role in determining the dynamics of how aggregates collide, and therefore also plays an important role in understanding the mechanisms involved in the formation of aggregates from collisions of other aggregates [1], [2]. The coagulation rate and fractal dimensions of aggregates formed in a dust plasma both depend on how the charge is distributed on the irregularly shaped aggregates. [3]

A plasma containing conducting dust grains is a significantly different situation from aggregates of non-conducting dust grains, because in conducting aggregates the charge flows freely in an effort to equalize the potential over the surface of the aggregate. We have approximately solved the problem of finding the charge distribution on a conducting aggregate of arbitrary shape and size in electrostatic equilibrium using numerical methods. The results of this research will be the focus of this paper.

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## II. Methods

### A. Preliminary Assumptions and Approximations

In this approximate solution, we operated under the assumption that the dust grains were spherical in shape and had a common radius. We also assumed that the charge on each dust grain was evenly distributed among the surface of that particular grain in a spherically symmetric manner. In reality, it is not true that the charge on each grain will be spherically symmetric, because this sort of charge distribution will not equalize the potential on the surface of the entire aggregate.

Even so, we expect that the effects of variations in the local charge on a grain will not cause a significant difference between our simulations and the behavior of a physical aggregate. As for the assumption that the grains must be equally sized spheres, we consider simulations with ellipsoidal and otherwise non-spherical dust grains, as well as with grains of heterogeneous sizes, to be topics of future research.

### B. The Pincus Algorithm

We began by producing simulated aggregates using the method described by Lorin Matthews and Truell Hyde [1]. These aggregates were simple collections of spheres in contact with each other in an arrangement emulating the structure of

real aggregates. Each of these spheres was given an initial charge. We realized from first principles of electromagnetic theory that the charge on a conducting aggregate of this sort must redistribute itself in such a manner that the potential may be equalized throughout the surface of the aggregate.

Suppose then, that we have simulated an aggregate composed of $n$ spherical conducting grains. Suppose also that we select $N$ reference points on the surface of each grain (we selected these reference points in a spherically symmetric manner, though it is conceivable that they could be selected to be at random locations around the spheres). If this aggregate is in electrostatic equilibrium, then the reference points should be at a common potential. As such, the sum of the absolute values of the potential differences between each pair of these points should be zero. Therefore, we desired an algorithm to minimize this sum as a function of the charge distribution on the aggregate.

To find the charge distribution at which this sum is at its lowest possible value, we implemented the Monte Carlo method described by Martin Pincus [4]. Consider a real-valued, continuous function $f(x_1, x_2, \dots, x_m)$, where $x_1, x_2, \dots, x_m \in \mathbb{R}$, and a closed, bounded region $S \subseteq \mathbb{R}^m$. Suppose that there exists a unique point $x \in S$ such that $\forall y \in S\big(f(y) \geq f(x)\big)$. In that case, the algorithm described in the aforementioned paper can be used to approximately find this point $x$. As it is, the sum that we referred to in the previous paragraph is a continuous and real valued function of the charges on the grains, and we know that a unique minimizing point of the sum must exist, since the equilibrium charge distribution is unique.

### *C. The sum function $f(q_1, q_2, \dots, q_n)$*

We begin by introducing some notation that we will use throughout this paper. Let us assume that our aggregate is composed of dust grains numbered $1, 2, \dots, n$. For any $i \in \{1, 2, \dots, n\}$, the reference points on grain $i$ will be denoted as $i_1, i_2, \dots, i_N$. We denote the position vector of the center of mass of grain $i$ as $\overrightarrow{\boldsymbol{r_i}}$. We denote the position vector of the $j$th point on grain $i$ as $\overrightarrow{\boldsymbol{r_{i_j}}}$. Then the potential of the $j$th point on grain $i$ as a function of the grain charges, which we will denote $V_{i_j}(q_1, \dots, q_n)$, is given by:

$$V_{i_j}(q_1, \dots, q_n) = \frac{kq_1}{\left\|\overrightarrow{\boldsymbol{r_1}} - \overrightarrow{\boldsymbol{r_{i_j}}}\right\|} + \cdots + \frac{kq_n}{\left\|\overrightarrow{\boldsymbol{r_n}} - \overrightarrow{\boldsymbol{r_{i_j}}}\right\|}. \qquad (1)$$

We know that the potential must take this form, by the principle of superposition and by the corollary to Gauss' Law that implies that the electric field due to a spherical charge distribution is of the same form as that of a point charge.

That said, the function that we must minimize, the sum of the absolute values of the potential differences between the pairs of reference points on the aggregate, can be expressed as follows. Let there be some injective and surjective function $\sigma: \{1,2, \dots, nN\} \to \left\{V_{i_j} \middle| i \in \{1, 2, \dots, n\} \text{ and } j \in \{1, 2, \dots, N\}\right\}$, such that $\sigma(1) \geq \sigma(2) \geq \cdots \geq \sigma(nN)$. Then the function that we wish to minimize can be given by

$$\begin{aligned} f(q_1, q_2, \dots, q_n) = {} & \big(\sigma(1) - \sigma(2)\big) + \big(\sigma(1) - \sigma(3)\big) \dots \\ & + \big(\sigma(1) - \sigma(nN)\big) + \big(\sigma(2) - \sigma(3)\big) \\ & + \big(\sigma(2) - \sigma(4)\big) + \cdots + \big(\sigma(2) - \sigma(nN)\big) \\ & + \cdots + \big(\sigma(nN-1) - \sigma(nN)\big). \end{aligned} \qquad (2)$$

We know that this is the sum of the absolute values of the potential differences between the points on the aggregate because all of the terms of this sum are positive, by definition of $\sigma$. We note that this function can be simplified:

$$\begin{aligned} f(q_1, q_2, \dots, q_n) = {} & \left((nN-1)\sigma(1) - \sum_{i=2}^{nN} \sigma(i)\right) \\ & + \left((nN-2)\sigma(2) - \sum_{i=3}^{nN} \sigma(i)\right) + \cdots \\ & + \left(\sigma(nN-1) - \sum_{i=nN}^{nN} \sigma(i)\right), \end{aligned} \qquad (3)$$

and further,

$$f(q_1, q_2, \dots, q_n) = \sum_{i=1}^{nN} (nN - 2i + 1)\, \sigma(i). \qquad (4)$$

This, then, is the form of the function that we seek to minimize using the Pincus Algorithm.

### *D. The state space*

This brings us to a new question: what will be this closed and bounded domain $S \subseteq \mathbb{R}^n$? The answer to this question lies in the fact that the number of possible charge distributions of an aggregate with a given charge is finite, because charges are discrete. If we had assumed that any fraction of a Coulomb could be transferred between grains, then the same condition would not hold, since then the number of possible charge distributions would, indeed, be infinite. Knowing, then, that the set of all possible charge distributions, which we will denote by $\mathfrak{Q}(n, a)$ (where $n$ is the number of grains in the aggregate and $a$ is the number of elementary charges in the aggregate), is finite, we considered that we may use it as our state space $S$. Note that, strictly speaking, $\mathfrak{Q}(n, a)$ is a set which is the output of a function; it is defined rigorously by the function $\mathfrak{Q}: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathcal{P}(\mathbb{Z}^+)$ such that

$$\mathfrak{Q}(n, a) = \left\{\{e_1, e_2, \dots, e_n\} \subseteq \mathbb{Z}^+ \middle| e_1 + e_2 + \cdots + e_n = a\right\}. \qquad (5)$$

This approach of using $S = \mathfrak{Q}(n, a)$ is not possible for arbitrary cases. Consider that, for an aggregate with $a$ electrons distributed among $n$ dust grains, the size of $\mathfrak{Q}$ is

$$|\mathfrak{Q}(n, a)| = \frac{(a + n - 1)!}{a!\ (n-1)!}. \qquad (6)$$

(Note that this equation is the answer to the combinatorial problem of "if I have $a$ non-distinct apples and $n$ barrels of infinite volume, in how many different ways can I distribute the apples among some or all of the barrels?" [5]) It is typical for us to have about $10^4$ electrons on each grain. Suppose then, that we have an aggregate of three dust grains with that amount of charge initially present on each grain. In that case, $|\mathfrak{Q}(n,a)| \approx 4.5(10^8)$. Already, this number is quite large for a modern computer to handle. If we then consider aggregates of more grains than just three, we see that the situation quickly goes out of control.

It was necessary, therefore, for us to find a different method of using a state space of possible charge distributions. Rather than considering all statistical possibilities, we chose instead to consider $S$ to be the set of all physically sensible charge distributions. What charge distribution, then, is physically sensible? Since the aggregate is composed of conducting material, it is sensible to assume that the charge will be concentrated at those grains on the edges of the aggregate. If this were not true, then the charge would be distributed in such a way that the electrons are uncomfortably juxtaposed on interior grains while also repelling each other. The charges, then, will tend to spread out as much as possible. Since most of the members of $\mathfrak{Q}(n,a)$ do not satisfy this rather particular condition, they can be excluded as obviously not the equilibrium charge distribution and therefore not to be considered in $S$.

But there are subtleties to consider here; aggregates are discrete structures formed from finite numbers of dust grains and therefore cannot be thought of in the same way as metallic solids, where the edge is clearly defined. In fact, defining what is meant by the term "edge" in a discrete structure of spherical dust grains is greatly difficult. In addition, while most of the charge will be concentrated around the edge grains, it is not prudent to assume in the general case that the interior grains will have no charge at all; some grains behave like edges more than others do, and the charge distribution becomes biased accordingly. We have introduced the term "edge of the aggregate" in this paper for the sole purpose of examining results provided by the program, and we acknowledge that the term possesses some degree of subjectivity.

Therefore, since we cannot so easily determine what to do about large aggregates from first principles, we considered aggregates of ten or less grains first. If we assume that the elementary charge is three hundred electron charges (that is, that electrons move in groups of three hundred), then for an aggregate of three dust grains and ten thousand electrons per grain, the number of possible charge distributions swiftly drops from $4.5(10^8)$ to a reasonable $5151$. This makes the cardinality of our state space suddenly reasonable and gives a rather small error in the charge on any particular grain. (This error will be discussed in detail later.) Therefore, with small aggregates, it is possible for us to use $S = \mathfrak{Q}(n,a)$ as long as we assume that the elementary charge is some integer number $\mathfrak{U} \geq 1$ of electron charges. For the rest of this paper, I will call the effective elementary charge, $\mathfrak{U}e$, "the charge on one aplectron" (the neologism is a portmanteau of "approximate" and "electron"). For the rest of this paper, $a$ will denote the number of aplectrons present in an aggregate.

We simulated aggregates of ten or less grains in hopes that we would notice a trend in the data that would give us some insight about larger aggregates. But before discussing the results of these simulations, we turn our attention to a subtlety pertaining to the previous paragraph. Our computers can comfortably handle generating state spaces with cardinalities of up to about thirty million for ten aggregates. But if we make the same assumption as before, that is, that $\mathfrak{U} = 300$, and attempt to simulate an aggregate of nine dust grains with $10^4$ electrons per grain, we will have $\mathfrak{Q}(9{,}300) = 1.83(10^{15})$ different charge distributions. But if $\mathfrak{U} = 3500$, then the $\mathfrak{Q}(9{,}26)$ is the much more reasonable $16.8(10^6)$ (we have used $26$ rather than the actual value of $\frac{90000}{3500}$ because we must have that the number of aplectrons be integral). Therefore, we designed the program so that the amount of charge of one aplectron must be determined dynamically based on the number of electrons present in the aggregate. One may say that it may be sufficient to let $\mathfrak{U} = 3500$ for all aggregates, but that would provide blatantly inaccurate results for aggregates that have, for example, only $3(10^4)$ electrons on them, since the error in the charge on each monomer would be so great.

## III. Results

At this time, our results are not complete; our future work on this project will provide much more accurate and reliable results than we currently have. Our current results are impeded by some limitations that have resulted from the amount of time allotted to this NSF-funded program. Therefore, better results are forthcoming. A sample of our preliminary results is presented in the form of the effect of the program on an aggregate of six dust grains.

First of all, in future work, we intend to dynamically determine the number of aplectrons based on the number of electrons in the aggregate so that the cardinality of the state space will be close to $16.8(10^6)$. In these preliminary simulations, we did not implement this. It turns out that most dust grains contain several thousand less than ten thousand electrons; no aggregate that we have previously simulated has contained a dust grain with ten thousand or more electrons in the initial state (although many grains have had numbers of electrons that are rather close to ten thousand). In fact, we have considered that the grains will have approximately ten thousand electrons each as a worst-case scenario in order to calculate upper bounds on the cardinalities of the state space. Since many of the grains did not have numbers of electrons that we close to ten thousand, it turned out that the aggregate had many less electrons than we had predicted. As a result, we could not predict, *a priori*, what the size of an aplectron should be. It happened that $\mathfrak{U} = 3$ provided somewhat accurate results, so that was used to produce the results featured below.

Second, the cardinality of the state space was not close to $16.8(10^6)$, but rather was chosen (for reasons involving the

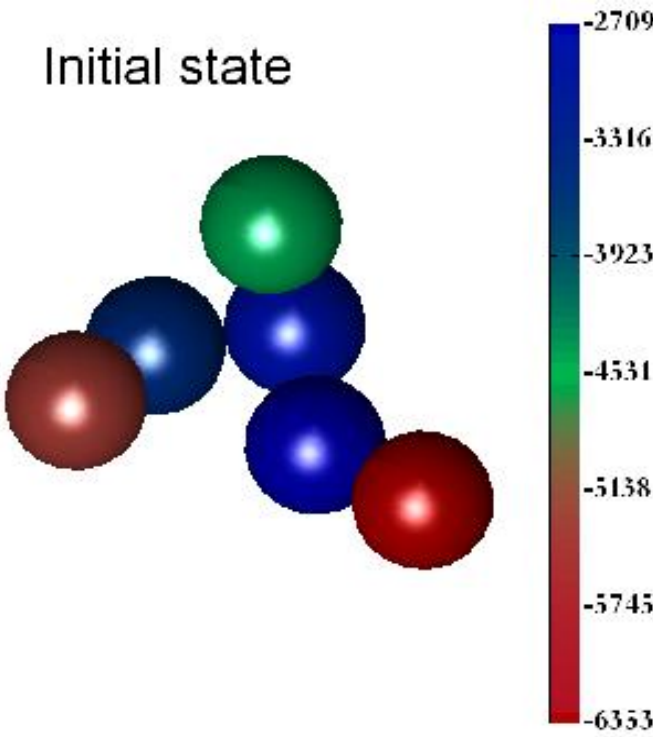


Fig. 1. The charge distribution on an aggregate of conducting dust grains in the initial state. This is the charge distribution on the aggregate before the program acts on it. The scale is in terms of elementary charges.

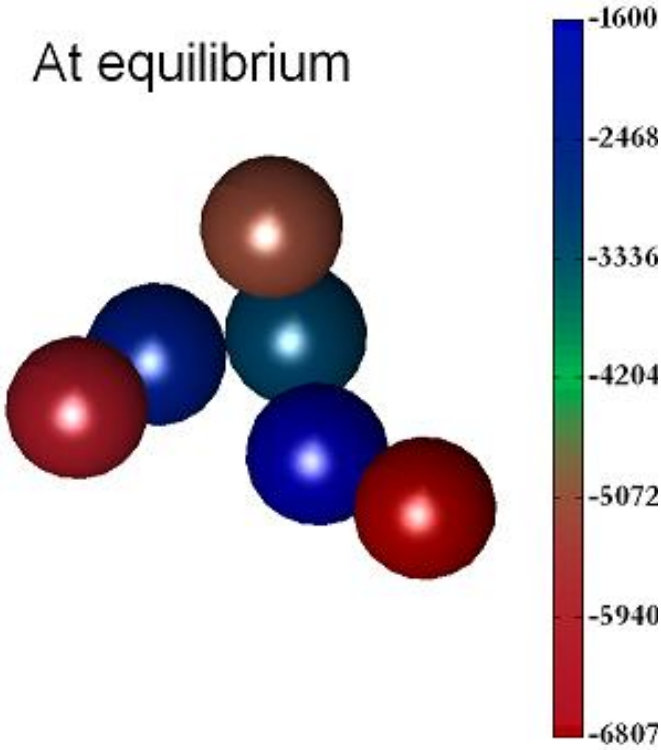


Fig. 2. The charge distribution on an aggregate of conducting dust grains in electrostatic equilibrium, according to our current simulations. The scale is in terms of elementary charges. Note that the color scale is different from that in Figure 1.

time required for the program to run) to be exactly $10^5$. In light of the fact that $\mathfrak{U} = 3$ in this simulation, this state space, then, is a rather small subset of the state space that would have been appropriate for an actual simulation. Indeed, the state space that would have been appropriate for this number of aplectrons with this many dust grains may even be too large for a modern computer to handle, in consideration of how quickly the function displayed in equation (6) rises as the number of dust grains is changed. (For the number of dust grains in the aggregate will inevitably change the number of electrons present in the entire aggregate. Our previous assumption that there are about $10^4$ electrons on each grain may have been inappropriate in the context of an actual simulation, and so $a \approx 10^4 n$ is not correct, but that does not mean that $a$ and $n$ are mutually exclusive, or even that their relationship is not approximately linear.)

The results are also dubious because of the calculus involved in the Pincus Algorithm. Notice in Martin Pincus' paper [4] that a limit $\lim_{\lambda \to \infty} r$ must be taken for some variable $r$. Of course this is not literally possible for a computer program to simulate with the sort of accuracy that is involved in calculus. Yet, the fact that our $\lambda$ is finite is not a significant problem; any computational simulation will have this issue, and it is not a source of concern for us. Rather, the issue is brought into light by the finiteness of $\lambda$. We used a logical "for loop" in our program in order to simulate this limit; we did the computation with one value of $\lambda$, and then did it again with a large value, and so on, until $\lambda$ became very large. But there is an exception to this in our program; if the same charge distribution is found for thirty different values of $\lambda$ consecutively, then the program will not iterate $\lambda$ anymore, but rather, will select the charge distribution that was chosen thirty times, since it is obvious at that point that changing $\lambda$ will have no effect on which charge distribution is chosen; it is large enough at that point. We have the following problem with this: the program, as it is currently written, iterates $\lambda$ only twice before finding the same charge distribution thirty times. This is uncomfortable, since it may mean that the program is choosing a charge distribution that provides a local minimum to the function in (4), rather than the absolute minimum of the function.

Yet even in light of the mediocrity of these preliminary results in comparison to the results we will get from our future research, these preliminary results seem to support our original hypothesis. As previously noted, the charge should bias toward the "edge" of the aggregate; it ought to be concentrated on those grains that are at the extreme points of the aggregate. While we cannot define this precisely, the reader should have some concept of what is meant by the extreme points on the aggregate. It is clear from Figure 2 that the charge is obeying this prediction. We think that it is important to emphasize that the difference between the red and blue in Figure 1 is smaller than the difference between the red and blue in Figure 2, as can be seen from the difference in the scales. So while these results do hold some degree of promise in terms of our original hypothesis, they are only precursors to the much more accurate results that we will be collecting soon.

## IV. Acknowledgments

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