Directions: Do exactly nine problems. At least four must be chosen from Part 1 and at least four must be chosen from Part 2.

Part 1. Theorems from class:

(a) Every compact subspace of a Hausdorff space is closed.

(b) If $X$ is path connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

(c) Let $p : E \to B$ be a covering map and let $p(\epsilon_0) = b_0$. Any path $f : [0, 1] \to B$ beginning at $b_0$ has a lifting to a path $\tilde{f}$ in $E$ beginning at $\epsilon_0$. [Note: You can skip uniqueness of the lift.]

(d) Let $X$ be path connected. Attach a 2-cell $B^2$ to $X$ by the map $\varphi : S^1 \to X$. In other words, let $Y = (X \coprod B^2) / \sim$ where $z \sim \varphi(z)$ for $z \in S^1$. Fix base points $x_0 \in X$ and $s_0 \in S^1 \subseteq B^2$ and let $\gamma$ be a path from $x_0$ to $\varphi(s_0)$. By abuse of notation, write $\varphi$ for the loop in $X$ generated by $\gamma$ (technically, the loop is given by $t \mapsto \varphi(e^{2\pi it})$). Let $N$ be the normal subgroup generated by $\tilde{\gamma}[\varphi]$. Show the inclusion map $i : X \to Y$ induces an isomorphism

$$\pi_1(X, x_0)/N \cong \pi_1(Y, x_0).$$

(e) If $\{X_\alpha\}$ is the collection of path connected components of $X$, and if $T_\alpha$ is a fixed singular 0-simplex with image in $X_\alpha$, then the homology classes $[[T_\alpha]]$ form a basis for $H_0(X)$.

(f) (Main Lemma for Homotopy Invariance) There exists, for each space $X$ and each non-negative integer $p$, a homomorphism

$$D_X : S_p(X) \to S_{p+1}(X \times I)$$

having the following properties:

1. If $T : \Delta_p \to X$ is a singular simplex, then

$$\partial D_X T + D_X \partial T = j_\#(T) - i_\#(T)$$

where $i(x) = (x, 0)$ and $j(x) = (x, 1)$.

2. $D_X$ is natural; that is, if $f : X \to Y$ is continuous, then the following diagram commutes:

$$\begin{array}{ccc}
S_p(X) & \to & S_{p+1}(X \times I) \\
\downarrow & & \downarrow \\
S_p(Y) & \to & S_{p+1}(Y \times I).
\end{array}$$

(g) (Main Lemma for Excision) Suppose $\mathcal{A}$ is a collection of subsets of $X$ whose interiors cover $X$. Then the inclusion map $S^A(X) \to S(X)$ induces an isomorphism in homology.
Part 2. Homework-like problems:

(a) For \( n \in \mathbb{N} \), let \( x_n = (x_{n,\alpha})_{\alpha \in \mathcal{J}} \) be a sequence in \( X = \prod_{\alpha \in \mathcal{J}} X_\alpha \) where each \( X_\alpha \) is a topological space and \( X \) has the product topology.

1. Show \( x_n \) converges to \( x = (x_\alpha) \in X \) if and only if, for each \( \alpha \), the sequence \( x_{n,\alpha} \in X_\alpha \) converges to \( x_\alpha \).
2. Show this statement can be false if \( X \) has the box topology.

(b) Let \( G \) be a topological group and \( H \) a subgroup of \( G \). Show that the map \( p : G \to G/H \) given by \( p(g) = gH \) is an open map.

(c) Calculate:
   1. \( \pi_1(\mathbb{R}P^2) \)
   2. \( \pi_1(T) \)
   3. \( \pi_1(X) \) where \( X \) is the 2-cell \( B^2 \) equipped with the equivalence relation \( z \sim z^3 \) on \( S^1 \subseteq B^2 \).
      In other words, wrap the edge around on itself three times.

(d) Compute \( H_1(K) \) and \( H_2(K) \) of the simplicial complex \( K \) given by:

(e) If there is a retraction \( r : K \to K_0 \), show

\[ H_p(K) \cong H_p(K, K_0) \oplus H_p(K_0). \]

(f) Let \( A \) be a closed subset of \( X \) and suppose \( A \) is a deformation retract of an open set in \( X \). If \( X/A \) is the space obtained by collapsing \( A \) to a point, show that

\[ H_p(X, A) \cong \tilde{H}_p(X/A). \]

(g) Using CW complexes, calculate the homology groups of \( X \) where:

1. \( X \) is made up of three open cells—one in dimension 0, one in dimension 2, and one in dimension 4.
2. \( X = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \)